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DYNAMICALLY STABLE PRINCIPLES OF OPTIMALITY IN COOPERATIVE DIFFERENTIAL GAMES ON QUICK ACTION OPERATION*

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The class of cooperative differential games on quick-action operation is determined, and the principles of optimality in it and questions of existence of solutions are considered. A superadditive characteristic set which is the analogue of a characteristic function in a cooperative game is constructed. The concept of sharing and of sharing predominance is introduced. The principle of dynamic stability is defined in a cooperative game on quick-action operation. The theorem on the existence of a dynamically stable c-nucleus is proved. The application of the proposed treatment of games of group pursuit on quick-action operation is considered.

1. Statement of the problem. The differential game of n players with dependent motions is considered. The dynamics of the game are defined by the set of equations

$$x^{\cdot} = f(x, u_1, \ldots, u_n),\ x \in R^m,\ u_i \in U_i \subset R^{m_i} \tag{1.1}$$

$$x(t_0) = x_0 \tag{1.2}$$

where U_i is a compact set of control parameters of the i-th player.

The admissible control of the i-th player is any measurable function $u_i(t)$ that satisfies in $[t_0, \infty)$ for any t the condition $u_i(t) \in U_i$.

It is assumed that system (1.1) has a unique solution $x(\cdot)$ continued in the half-interval $[t_0, \infty)$ for initial data $x_0 \in R^m$ and any set $(u_1(t), \ldots, u_n(t))$ of admissible controls. Moreover we assume that the vector function $f = (f_1, \ldots, f_m)$ on the right side of (1.1) can be represented in the form $f(x, u_1, \ldots, u_n) = f^1(x, u_1) + \ldots + f^n(x, u_n)$.

As admissible strategies of players we shall consider the piece-wise programme strategies (PPS). The (PPS) of the i-th player will be denoted by $u_i(\cdot)$ and the set of its (PPS) by D_i.

The game begins at the instant t_0 from the state x_0. The terminal sets $M_1, \ldots, M_n$ are specified in the phase space R^m. Let $(u_1(\cdot), \ldots, u_n(\cdot))$ be some admissible situation, and $x(\cdot) = x(\cdot, x_0, u_1(\cdot), \ldots, u_n(\cdot))$ be the trajectory of system (1.1)–(1.2) corresponding to that situation.

Definition 1. We call $T_i = T_i(x_0, u_1(\cdot), \ldots, u_n(\cdot))$ the frist instant when the phase point reaches the terminal set M_i in the situation $(u_1(\cdot), \ldots, u_n(\cdot))$, if $T_i = \min\{t \geqslant t_0 \mid x(t) \in M_i\}$.

Assumption A. The set of admissible situations $D_N = D_1 \times \ldots \times D_n$ is such that for any ordered sequence $M_{i_1}, \ldots, M_{i_n}$, the instants $T_{i_1}, \ldots, T_{i_n}$ exist and are finite.

Player i is interested in the phase point reaching the terminal set M_i in the shortest possible time, i.e. he aims to minimize the quantity

$$J_i(x_0, u_1, \ldots, u_n) = T_i(x_0, u_1, \ldots, u_n) - t_0 \tag{1.3}$$

Thus, the differential n-person game on quick-action operation has been defined in the normal form $\Gamma(x_0) = \langle x_0; D_1, \ldots, D_n; J_1, \ldots, J_n\rangle$.

2. The characteristic set. We denote by $N = \{1, \ldots, n\}$ the set of all players in the game $\Gamma(x_0)$. Any subset $S \subset N$, including the empty set $\varnothing$ and the set N itself, is called a coalition. Let the condition $S \subset N$ be formed. This means that members of the coalition S act as a single player with the set of strategies $D_S = \prod_{i \in S} D_i$ which aim to minimize the quantity J_i for all $i \in S$.

The vector $J = (J_1, \ldots, J_n)$, where J_i is the time taken to reach the terminal set M_i from the initial state x_0 (see (1.3)) is called the payoff vector. For each coalition $S \in 2^N$ we denote by $V(S, x_0)$ the set of all payoff vectors in the game $\Gamma(x_0)$, whose respective components the coalition S can guarantee to its members irrespective of the behaviour of the remaining players from the set $N \setminus S = \{i \in N \mid i \notin S\}$, including the case least favourable for S, when the coalition $N \setminus S$ works against it. The set $V(S, x_0)$ is the characteristic of the potential force of coalition S and is the basis of the definition of a cooperative game. Let us now construct the sets $V(S, x_0)$, $S \in 2^N$.

*Prikl.Matem.Mekhan.,50,1,3-16,1986

For the empty coalition $\varnothing$ we assume

$$V(\varnothing, x_0) = \varnothing \tag{2.1}$$

Let $S \subset N$ $(S \neq \varnothing)$. We fix the instant of time $\tau > t_0$, and consider the set of antagonistic differential games $\{\Gamma_S^y(x_0, \tau - t_0) \mid y \in R^m\}$ with a fixed duration $\tau - t_0$ between the coalitions S and $N \setminus S$. The dynamics of the game $\Gamma_S^y(x_0, \tau - t_0)$ are described by the equations

$$\dot{x} = f(x, u_S, u_{N\setminus S}),\ x(t_0) = x_0 \tag{2.2}$$

$$u_S \in \prod_{i\in S} U_i, \quad u_{N\setminus S} \in \prod_{j\in N\setminus S} U_j$$

The payoffs of the maximizing player has the form $K(x_0, u_S, u_{N\setminus S}) = -\rho(x(\tau), \dot{y})$, where $x(\tau)$, is the solution of system (2.2) at the instant τ, and ρ is the Euclidean distance. The payoff of $N \setminus S$ is equal to K. The value of the game $\Gamma_S^y(x_0, \tau - t_0)$ /2/

$$\operatorname{val}\Gamma_S^y(x_0, \tau - t_0) = \sup_{u_S(\cdot)\in D_S} \inf_{u_{N\setminus S}(\cdot)\in D_{N\setminus S}} K(x_0, u_S(\cdot), u_{N\setminus S}(\cdot)) = \inf_{u_{N\setminus S}(\cdot)\in D_{N\setminus S}} \sup_{u_S(\cdot)\in D_S} K(x_0, u_S(\cdot), u_{N\setminus S}(\cdot))$$

exists in the class of PPS.

Consider the set

$$Y_S^{\tau-t_0}(x_0) = \{y \in R^m \mid \operatorname{val}\Gamma_S^y(x_0, \tau - t_0) = 0\} \tag{2.3}$$

As implied by the value of game $\Gamma_S^y(x_0, \tau - t_0)$, for point $y \in Y_S^{\tau-t_0}(x_0)$ and preassigned $\varepsilon > 0$, the coalition S can guarantee the approach to y in a time $\tau - t_0$ to a distance not exceeding ε.

Let π be the transformation of every player i in πi and by the same token of each coalition $S = \{i_1, \ldots, i_s\}$ in the coalition $\pi S = \{\pi i_1, \ldots, \pi i_s\}$. Each sequence $\{M_i, i \in S\}$ of terminal sets arranged in the order in which they reach the coalition S, generates a transformation in the coalition S itself. Let the sequence

$$M_{j_1}, \ldots, M_{j_s}, \quad \text{where} \quad j_k = \pi i_k,\ k = 1, \ldots, s \tag{2.4}$$

correspond to it.

Let $T_{j_1}^S$ be the first instant of the set M_{j_1} is contacted by the set $Y_S^{\tau-t_0}(x_0)$ as τ increases, i.e.

$$T_{j_1}^S = \min\{\tau \mid Y_S^{\tau-t_0}(x_0) \cap M_{j_1} \neq \varnothing\}$$

If $Y_S^{\tau-t_0}(x_0) \cap M_{j_1} = \varnothing$, $\tau \geqslant t_0$, we assume that $T_{j_1}^S$ is equal to $+\infty$. To simplify the further calculations we shall make the following assumption.

Assumption B. When $T_1 = T_{j_1}^S$ is finite, there exists a unique point $x_1 = x(T_1)$ of first touching of the sets $Y_S^{\tau-t_0}(x_0)$ and M_{j_1}.

Then, similarly

$$T_2 = \min\{\tau \mid Y_S^{\tau-T_1}(x_1) \cap M_{j_2} \neq \varnothing\}, \ldots$$

$$T_s = \min\{\tau \mid Y_S^{\tau-T_{s-1}}(x_{s-1}) \cap M_{j_s} \neq \varnothing\}$$

If $T_k = +\infty$, then $T_l = +\infty$ for all $l = k + 1, \ldots, s$. This means than when "going around" the terminal sets in accordance with the transformation π, the sets $M_{j_k}, M_{j_{k+1}}, \ldots, M_{j_s}$ are not reached by coalition S. We thus obtain for π the sequence $T_{j_1}^S \leqslant \ldots \leqslant T_{j_s}^S$ or, which is the same, the sequence $T_{\pi i_1}^S \leqslant \ldots \leqslant T_{\pi i_s}^S$ of instants of reaching the sets (2.4).

We introduce the s-vectors $T_\pi^S = (T_{\pi i_1}^S, \ldots, T_{\pi i_s}^S)$, and $t_0 = (t_0, \ldots, t_0)$ and assume

$$V(S, x_0) = \{T_\pi^S - t_0 \mid \pi \in \pi_S\},\ S \subset N\ (S \neq \varnothing) \tag{2.5}$$

where π_S is the set of all transformations of the terms of coalition S. The quantity $T_{\pi i_k}^S - t_0$ (the k-th component of the vector $T_\pi{}^S - t_0$) has the meaning of the time of reaching the set M_{i_k} $(i_k \in S)$ by the coalition S under conditions of transformation π. Note that for $S \subset N$ $V(S, x_0)$ is a subset of space R^s, and its power equals $s!$, where s is the number of players in coalition S.

Let η be some s-vector. If $\xi \in V(S, x_0)$ and $\eta_i \geqslant \xi_i$ for all $i \in S$, we shall assume that $\eta \in V(S, x_0)$. We define the inclusion $\subset$ as follows. If for any $\eta \in A$ there exists a $\xi \in V(S, x_0)$, such that $\eta_i \geqslant \xi_i$, $i \in S$, then $A \mathrel{\tilde{\subset}} V(S, x_0)$.

Let $S, R \subset N$, $S \cap R = \varnothing$. Let us consider the direct product

$$V(S, x_0) \times V(R, x_0) = \{(\xi, \eta) \mid \xi \in V(S, x_0), \eta \in V(R, x_0)\}$$

which is a subset of space R^{s+r}, where $s(r)$ is the number of players in $S(R)$. The advantage of the coalition $S \bigcup R$ over the coalition S and R, if it exists, can be shown by the relation

$$V(S \bigcup R, x_0) \supseteqq V(S, x_0) \times V(R, x_0) \tag{2.6}$$

If (2.6) exists for all $S, R \subset N$, $S \bigcap R = \varnothing$, we shall call the set $V(S, x_0)$ super additive with respect to S.

Lemma 1. The set V defined by (2.1) and (2.5) is superadditive with respect to S.

Proof. The sets $V(S \bigcup R, x_0)$ and $V(S, x_0) \times V(R, x_0)$ are subsets of one and the same space R^{s+r} hence the relation $\supseteqq$ between the two is determined correctly. Moreover, it is evident that $(s+r)! > s!\, r!$ and $\pi_S \times \pi_R \subset \pi_{S \bigcup R}$, where $\pi_{S \bigcup R}$ is the set of all transformations in $S \bigcup R$. We set $S = \{i_1, \ldots, i_s\}$, $R = \{j_1, \ldots, j_r\}$. Let $\xi \in V(S, x_0) \times V(R, x_0)$ which indicates the existence of transformations $\varphi \in \pi_S$ and $\psi \in \pi_R$, for which

$$\xi = (T^S_{\varphi i_1} - t_0, \ldots, T^S_{\varphi i_s} - t_0,\ T^R_{\psi j_1} - t_0, \ldots, T^R_{\psi j_r} - t_0)$$

We arrange the instants $t_0, T^S_{\varphi i_1}, \ldots, T^S_{\varphi i_s}, T^R_{\psi j_1}, \ldots, T^R_{\psi j_r}$ in increasing order, and obtain the sequence

$$t_0 < T_1 \leqslant T_2 \leqslant \ldots \leqslant T_{s+r} \leqslant \infty \tag{2.7}$$

to which corresponds the sequence of points

$$\{x_k = x(T_k),\ k = 1, \ldots, s+r\} \tag{2.8}$$

$$x(T_k) = \begin{cases} x(T^S_{\varphi i_p}),\ T_k = T^S_{\varphi i_p} \\ x(T^R_{\psi j_q}),\ T_k = T^R_{\psi j_q} \end{cases}$$

where $x(T^S_{\pi k})$ is the point of first touching of the sets $Y_S^{\tau - T^S_{\pi k-1}}(x(T^S_{\pi k-1}))$ and $M_{\pi k}$. The sequence (2.8) induces a certain transformation $\bar{\pi} \in \pi_{S \bigcup R}$. With the transformation $\bar{\pi}$ the vector ξ is transformed into the vector $\bar{\pi}\xi = (T_1 - t_0, \ldots, T_{s+r} - t_0)$.

Consider the coalition $S \bigcup R$, and assume that the sets M_i, $i \in S \bigcup R$ are ordered (are reached) according to the rearrangement $\bar{\pi}$:

$$M_{l_1}, \ldots, M_{l_{s+r}} \tag{2.9}$$

We will put $\eta = (T_1^{S \bigcup R} - t_0, \ldots, T_{s+r}^{S \bigcup R} - t_0)$, where $T_k^{S \bigcup R}$ is the first instant, when the set M_{l_k}, $l_k \in S \bigcup R$ is reached as a result of the transformation $\bar{\pi}$. To prove the lemma it is sufficient to show that

$$\eta_k \leqslant \bar{\pi}\xi_k,\ k = 1, \ldots, s+r \tag{2.10}$$

For the transformation $\bar{\pi}$ the points (2.8) are generally not points that first touch the sets (2.9), but to prove the inequalities (2.10) it is sufficient to show that the points (2.8) will be reached by the coalition $S \bigcup R$ in a time not exceeding $T_1 - t_0, \ldots, T_{s+r} - t_0$ respectively.

Let x_k be an arbitrary point of the sequence (2.8). We assume that for $i \in S$ we have $x_k \in M_i$ (for $i \in R$ the reasoning is similar). Then by definition x_k is the point of first touching of the sets $Y_S^{T_k - T_{k-1}}(x(T_{k-1}))$ and M_i. Here $T_k = T^S_k$, since $x_k \in M_i$ for $i \in S$, but not always $T_{k-1} = T^S_{k-1}$, since $x_{k-1} \in M_j$ is possible, where $j \in R$. The time of conversion of the phase point from the state x_{k-1} to x_k by the efforts of the coalition S is equal to $T_k - T_{k-1}$. To this time there corresponds a pair of ε-optimal strategies $(\bar{u}^\varepsilon_S(\cdot), \bar{u}^\varepsilon_{N \setminus S}(\cdot))$ of the game $\Gamma_S^{x_k}(x_{k-1}, T_k - T_{k-1})$ (see (2.3)).

Consider now the game $\Gamma^{x_k}_{S \bigcup R}(x_{k-1}, T_k^{S \bigcup R} - T_{k-1}^{S \bigcup R})$, where $T_k^{S \bigcup R} - T_{k-1}^{S \bigcup R}$ is the time of conversion of the phase points from the state x_{k-1} to the state x_k by the efforts of the coalition $S \bigcup R$. The following inequality holds:

$$T_k - T_{k-1} \geqslant T_k^{S \bigcup R} - T_{k-1}^{S \bigcup R} \tag{2.11}$$

Indeed, let us construct a strategy $u'_{S \bigcup R}(\cdot) = \{\bar{u}^\varepsilon_S(\cdot),\ \bar{u}^\varepsilon_{N \setminus S}(\cdot)|_R\}$, where $\bar{u}^\varepsilon_S(\cdot)$ is the ε-optimal strategy of the coalition S in the game $\Gamma_S^{x_k}(x_{k-1},\ T^S_k - T^S_{k-1})$, and $\bar{u}^\varepsilon_{N \setminus S}(\cdot)|_R$ is the truncation on R of the ε-optimal strategy of the coalition $N \setminus S$ in that game. It is obvious that $u'_{S \bigcup R}(\cdot) \in D_{S \bigcup R}$ and that, applying this strategy, the coalition $S \bigcup R$ transforms the phase point from the state x_{k-1} to the state x_k in a time not exceeding $T_k - T_{k-1}$. Moreover, applying the strategy $\bar{u}^\varepsilon_{S \bigcup R}(\cdot)$, which is ε-optimal in the game $\Gamma^{x_k}_{S \bigcup R}(x_{k-1},\ T_k^{S \bigcup R} - T_{k-1}^{S \bigcup R})$, the coalition $S \bigcup R$ transforms the phase point from the state x_{k-1} to the state x_k in a time $T_k^{S \bigcup R} - T_{k-1}^{S \bigcup R}$, not exceeding $T_k - T_{k-1}$.

From (2.11) with $k = 1$ we obtain

$$T_1 - t_0 \geqslant T_1^{S \bigcup R} - t_0 \tag{2.12}$$

i.e. $\eta_1 \leqslant \bar{\pi}\xi_1$. From (2.11) with $k=2$, using (2.12), we obtain $T_2^{S \cup R} \leqslant T_2$, i.e. $\eta_2 \leqslant \bar{\pi}\xi_2$. Continuing this for $k=3,\ldots,s+r$, we obtain (2.10). Then, since $\eta \in V(S \cup R, x_0)$, it follows from (2.10) that $\bar{\pi}\xi \in V(S \cup R, x_0)$. Because of the arbitrariness of the transformations of φ and ψ, that generate the transformation $\bar{\pi}$, we obtain that $\xi \in V(S \cup R, x_0)$. Hence relation (2.6) holds.

The superadditive set V is called the characteristic set. The determination for each coalition $S \in 2^N$ of the characteristic set $V(S, x_0)$ means the determination of the cooperative game $\Gamma_V(x_0) = \langle N, V(S, x_0)\rangle$. The aim of players in the game $\Gamma_V(x_0)$ is to minimize the time taken to reach the terminal sets; hence we call the game $\Gamma_V(x_0)$ the cooperative differential game on quick-action operation.

3. The principle of dynamic stability of solutions in the game $\Gamma_V(x_0)$. First we introduce the concept of sharing and of predominance of sharing in the game $\Gamma_V(x_0)$.

Definition 2. Any vector $\xi \in R^n$, that satisfies the conditions: 1) for all $i \in N$ we have $\xi_i \leqslant V(\{i\}, x_0)$; 2) $\xi \in V(N, x_0)$, is called the sharing in the game $\Gamma_V(x_0)$.

We denote the set of all sharings in the game $\Gamma_V(x_0)$ by $E_V(x_0)$. It is clear that $E_V(x_0) \subset V(N, x_0)$.

Let $\xi \in E_V(x_0)$, and $\xi^S = \{\xi_i, i \in S\}$, i.e. ξ^S is an s-dimensional vector composed of components of sharing ξ that correspond to S.

Definition 3. We say that the sharing ξ predominates over the sharing η by the coalition S ($\xi \succ_S \eta$), if 1) for all $i \in S$ we have $\xi_i < \eta_i$, and 2) $\xi^S \in V(S, x_0)$. We say that the sharing ξ predominates over the sharing η ($\xi \succ \eta$), if a coalition $S \subset N$ is found such that $\xi \succ_S \eta$. Predominance is not possible over coalitions consisting of one player. Actually from $\xi \succ_i \eta$ it follows that $\eta_i > V(\{i\}, x_0)$ which is impossible (see Definition 2)). Note that predominance over N is possible.

Predominance in the sense of Definition 3 can be used to define the c-kernel, the NM solution and other concepts of the solutions of the game $\Gamma_V(x_0)$, as is done in classical cooperative theory /3/.

Let $W_V(x_0) \subset E_V(x_0)$ be some solution of the game $\Gamma_V(x_0)$ determined for the state x_0. Each sharing $\xi = (\xi_1, \ldots, \xi_n)$ represents the time of reaching in some definite way the ordered sequence $M_{i_1}, \ldots, M_{i_n}$ of terminal sets. Consequently, to each sharing $\xi \in E_V(x_0)$ there corresponds a trajectory $x(\cdot)$ of system (1.1)–(1.2) such that $\xi_k = T_{i_k}^N(x(\cdot)) - t_0$, where $T_{i_k}^N(x(\cdot))$ is the instant of reaching the set M_{i_k}, when moving along the trajectory $x(\cdot)$.

Definition 4. Let $W_V(x_0) \neq \varnothing$. We shall call any trajectory $x(\cdot)$ of system (1.1)–(1.2) such that $[T^N(x(\cdot)) - t_0] \in W_V(x_0)$ the conditionally optimal trajectory. Here $T^N(x(\cdot)) - t_0 = (T_{i_1}^N(x(\cdot)) - t_0, \ldots, T_{i_n}^N(x(\cdot)) - t_0)$.

We will now formulate the principle of dynamic stability in the game $\Gamma_V(x_0)$. Note that for cooperative games with transferable payoffs the concept of dynamic stability was introduced in /1/, and for games with non-transferable payoffs in /4, 5/.

Consider the games in progress $\Gamma_V(x(t))$ and their solutions $W_V(x(t)) \subset E_V(x(t))$ along the conditionally optimal trajectory $x(\cdot)$ Let $\xi^t \in W_V(x(t))$. The component ξ_i^t of the sharing ξ^t is the time taken to reach the set M_i from the state $x(t)$ when $i \in N$. It will be seen that $\xi_i^t = 0$ for all i such that $T_i^N(x(\cdot)) \leqslant t$. We put $\bar{T} = \max_{i \in N} T_i^N(x(\cdot))$.

Definition 5. Suppose $\xi \in W_V(x_0)$, and $x(\cdot)$ is the conditionally optimal trajectory such that $T^N(x(\cdot)) - t_0 = \xi$. The sharing ξ is called dynamically stable, if $W_V(x(t)) \neq \varnothing$ for all $t_0 \leqslant t \leqslant \bar{T}$, and

$$\xi \in \bigcap_{t_0 \leqslant t \leqslant \bar{T}} [\tau(t) + W_V(x(t))]$$

$$\tau(t) = (\tau_1(t), \ldots, \tau_n(t)); \quad \tau_i(t) = \min\{\xi_i, t - t_0\}, \quad i \in N$$

In this case the conditionally optimal trajectory is called optimal.

The solution $W_V(x_0)$ is called the dynamically stable solution of the game $\Gamma_V(x_0)$, if all sharings $\xi \in W_V(x_0)$ are dynamically stable.

In Definition 5 the sum $\tau(t) + W_V(x(t))$ is a set of vectors of the form $\tau(t) + \xi^t$ where $\xi^t \in W_V(x(t))$. Consequently, for the dynamically stable sharing $\xi \in W_V(x_0)$ a sharing $\xi^t \in W_V(x(t))$ can be found at every instant of time $t \in [t_0, \bar{T}]$ such that $\xi = \tau(t) + \xi^t$.

The principle of dynamic stability is the principle of realizability of sharing in the differential game. It has the important property that the agreement reached by the players at the beginning of the game regarding the sequence and time of reaching terminal sets (sharing $\xi \in W_V(x_0)$) is maintained till the end of the game, when moving along the optimal trajectory, in other words, it will be realized.

4. The dynamically stable c-kernel. In what follows we shall understand the set $W_V(x_0)$ to mean the c-kernel of the game $\Gamma_V(x_0)$, and shall denote it by $C_V(x_0)$.

Definition 6. The set of all non-predominating (in the meaning of Definition 3) sharings is called the c-kernel of the game $\Gamma_V(x_0)$.

Definition 7. The game $\Gamma_V(x_0)$ is called N-substantial, if for all $S \subset N\ (S \neq N)$ and $\tau \geqslant t_0$ and the equation $Y_S^{\tau - t_0}(x_0) \cap M_i = \varnothing$ is satisfied for at least one $i \in S$.

Definition 7 and Assumption A imply that in the N-substantial game the terminal sets are reached only when the maximal coalition N is formed.

Theorem 1. In an N-substantial game $\Gamma_V(x_0)$ a non-empty c-kernel $C_V(x_0)$ exists.

Proof. It follows from Assumption A that the set $E_V(x_0)$ of sharings in an N-substantial game is non-empty. Let $\bar\xi, \bar{\bar\xi} \in E_V(x_0)$ We assume the existence of the coalition $S\ (S \neq N)$, such that $\bar\xi \underset{S}{\succ} \bar{\bar\xi}$. Then $\bar\xi_i < \bar{\bar\xi}_i$ for all $i \in S$ and $\bar\xi^S \in V(S, x_0)$ (see Definition 3). By the definition of the N-substantial game for all $\eta \in V(S, x_0)$ and $\eta_i = +\infty$ for at least one $i \in S$. Since $\bar\xi_i^S < \infty$ for all $i \in S$, in $V(S, x_0)$ does not contain a single vector η such that $\eta_i \leqslant \bar\xi_i^S$ for all $i \in S$, i.e. $\bar\xi^S \notin V(S, x_0)$, and, consequently $\bar\xi$ cannot predominate over $\bar{\bar\xi}$ in the coalition S. Consider now the coalition N. Two cases are possible 1) not a single sharing $\xi \in E_V(x_0)$ domainates over N and 2) a sharing exists which is domainted by N. In the first case $C_V(x_0) = E_V(x_0)$. In the second case at least one sharing exists that is not dominated by the coalition N, and $C_V(x_0) \subset E_V(x_0)$. Thus the c-kernel is in both cases non-empty. The theorem is proved

With the notation $\mathrm{dom}_N E_V(x_0) = \{\xi \in E_V(x_0)|$, we have $\eta \in E_V(x_0)$, $\eta \succ_N \xi\}$, $E_V(x_0) \setminus \mathrm{dom}_N E_V(x_0) = \{\xi \in E_V(x_0) |\ \xi \notin \mathrm{dom}_N E_V(x_0)\}$.

Corollary. In an N-substantial game $\Gamma_V(x_0)$ $C_V(x_0) = E_V(x_0) \setminus \mathrm{dom}_N E_V(x_0)$.

Theorem 2. In an N-substantial game $\Gamma_V(x_0)$ for each conditionally optimal trajectory $\bar x(\cdot)$, all current games $\Gamma_V(\bar x(t))$ are N-substantial, then all c-kernels $C_V(x_0)$ are dynamically stable.

Proof. Let the sharing $\xi = (\xi_{i_1}, \ldots, \xi_{i_n})$, where $\xi_{i_k} = T_{i_k}^N - t_0 = T_{i_k}^N(\bar x(\cdot)) - t_0, k = 1, \ldots, n$ belongs to the c-kernel. The ordered sequence $M_{i_1}, \ldots, M_{i_n}$ corresponds to sharing ξ such that ξ_{i_k} is equal to the time of reaching the set M_{i_k} by coalition N along the conditionally optimal trajectory $\bar x(\cdot)$.

The points $\bar x(T_{i_1}^N), \ldots, \bar x(T_{i_n}^N)$ are the points of first touching of the sets $Y_N^{T_{i_1}^N - t_0}(x_0), \ldots, Y_N^{T_{i_n}^N - T_{i_{n-1}}^N}(\bar x(T_{i_{n-1}}^N))$ and the sets $M_{i_1}, \ldots, M_{i_n}$.

Consider the time interval $[t_0, T_{i_1}^N]$. Since for all $t_0 \leqslant t \leqslant T_{i_1}^N$ we have $Y_N^{T_{i_1}^N - t}(\bar x(t)) \subset Y_N^{T_{i_1}^N - t_0}(x_0)$ and $\bar x(T_{i_1}^N) \in Y_N^{T_{i_1}^N - t}(\bar x(t))$, where $\bar x(T_{i_1}^N)$ is the unique point of touching between $Y_N^{T_{i_1}^N - t_0}(x_0)$ and M_{i_1} (see Assumptions A and B), hence the point $\bar x(T_{i_1}^N)$ remains the unique point of touching between the sets $Y_N^{T_{i_1}^N - t}(\bar x(t))$ and M_{i_1} for all $t_0 \leqslant t \leqslant T_{i_1}^N$. Since by the condition of the theorem the game $\Gamma_V(\bar x(t))$, $t \in [t_0, T_{i_1}^N]$, is N-substantial, the sharing $\xi^t = \{T_{i_k}^N - t,\ k = 1, \ldots, n\}$ belongs to the c-kernel $C_V(\bar x(t))$. From this

$$\xi = [(t - t_0) + \xi^t] \in C_V(x_0), \quad t_0 \leqslant t \leqslant T_{i_1}^N \qquad (4.1)$$
$$t = (\underbrace{t, \ldots, t}_{n}), \quad t_0 = (\underbrace{t_0, \ldots, t_0}_{n})$$

At the instant $t = T_{i_1}^N$ we have $\xi_{i_1}^t = 0$.

Consider now the time interval $[T_{i_1}^N, T_{i_2}^N]$. We similarly obtain that the inclusion (4.1) holds, where $\xi^t \in C_V(\bar x(t))$ and $\xi_{i_1}^t = 0$ for all $t \geqslant T_{i_1}^N$, and at the instant $t = T_{i_2}^N$ also $\xi_{i_2}^t = 0$. From this it follows that

$$\xi \in \bigcap_{t_0 \leqslant t \leqslant T_{i_2}^N} [\tau(t) + C_V(\bar x(t))]$$

$$\tau(t) = \begin{cases} (\underbrace{t - t_0, \ldots, t - t_0}_{n}), & t_0 \leqslant t \leqslant T_{i_1}^N \\ (\underbrace{T_{i_1}^N - t_0,\ t - t_0, \ldots, t - t_0}_{n-1}), & T_{i_1}^N \leqslant t < T_{i_2}^N \\ (T_{i_1}^N - t_0,\ T_{i_2}^N - t_0,\ \underbrace{t - t_0, \ldots, t - t_0}_{n-2}), & t = T_{i_2}^N \end{cases}$$

Continuing this reasoning to the instant $T_{i_n}^N$, we obtain

$$\xi \in \bigcap_{t_0 \leqslant t \leqslant T_{i_n}^N} [\tau(t) + C_V(\bar x(t))]$$

where $\tau(t) = (\tau_1(t), \ldots, \tau_n(t))$, $\tau_i(t) = \min\{\xi_i, t - t_0\}$. The theorem is proved.

5. The game of group pursuit on quick-action operation. The method outlined here is used below to determine and investigate the cooperative differential game of four players on quick-action operation in a formulation close to problems of simple pursuit /6/.

The game between the pursuers P_1, P_2 and pursued E_1, E_2 occurs in a plane. The players move at constant velocities (α_i for P_i and β_j for E_j) and can change the direction of motion at any instant of time. The motions of the pursuers are dependent and defined by the equations

$$x^{\cdot 1} = u_1^1 + u_2^1, \quad x^{\cdot 2} = u_1^2 + u_2^2 \tag{5.1}$$

$$(u_i^1)^2 + (u_i^2)^2 = \alpha_i^2, \quad i = 1, 2 \tag{5.2}$$

The equations of motion of the pursued players E_1 and E_2 have the form

$$y^{\cdot 1} = v_1^1, \quad y^{\cdot 2} = v_1^2 \tag{5.3}$$

$$z^{\cdot 1} = v_2^1, \quad z^{\cdot 2} = v_2^2 \tag{5.4}$$

$$(v_j^1)^2 + (v_j^2)^2 = \beta_j^2, \quad j = 1, 2 \tag{5.5}$$

Thus $u_i = (u_i^1, u_i^2)$ is the velocity vector of the pursuers P_i and $v_j = (v_j^1, v_j^2)$ of the pursued player E_j, and at each instant of time the position of the pursuants is represented by a single point $x = (x^1, x^2)$, and the position of the pursued E_1 and E_2 by the points $y = (y^1, y^2)$ and $z = (z^1, z^2)$ respectively. The motion of the players begin at the instant $t_0 = 0$ from initial positions

$$x(t_0) = x_0, \; y(t_0) = y_0, \quad z(t_0) = z_0 \tag{5.6}$$

that do not lie on one straight line.

The state of information in the game is as follows. At every instant of time the pursuer P_i knows his position and the position of player E_i and the direction of his velocity. The pursued player E_j knows his position and has information on the position of player P_j.

Let us determine the strategy of the players. The pair $(\Delta_i, u_i^{\Delta_i})$, where Δ_i is some partitioning $t_0^{\Delta_i} = 0 < t_1^{\Delta_i} < \ldots < t_k^{\Delta_i} < \ldots$ of the half-interval which does not contain finite concentration points, and $u_i^{\Delta_i}$ is any vector function that has values in the circle (5.2) is called the piecewise constant strategy PPS of player P_i. Similarly the PPS of player $E_1(E_2)$ consists of a pair (σ, v_1^σ) $((\mu, v_2^\mu))$, where $\sigma(\mu)$ of some finite partitioning of the time interval $[0, \infty)$ that has no finite concentration of points, and $v_1^\sigma(v_2^\mu)$ is any vector function that has values in the circle (5.5) for $j = 1$ $(j = 2)$.

The strategy PPS $(\Delta_1, u_1^{\Delta_1})$, $(\Delta_2, u_2^{\Delta_2})$, (σ, v_1^σ), (μ, v_2^μ) will be denoted simply by u_1, u_2, v_1, v_2. The programme controls $u_1(t), u_2(t), v_1(t), v_2(t)$, if they are chosen in the class of piecewise constant vector functions. These controls are called programmed strategies.

Let $x(\cdot, x_0, u_1, u_2)$, $y(\cdot, y_0, v_1)$, $z(\cdot, z_0, v_2)$ be the trajectories of system (5.1), (5.3), and (5.4) in the situation (u_1, u_2, v_1, v_2) that start from the initial states (5.6).

We call $T_1 = T_1(x_0, y_0; u_1, u_2, v_1)$ the instant of encounter of players P_1 and E_1 with strategies (u_1, u_2, v_1), if

$$T_1 = \min\{t \geqslant t_0 \mid x(t) = y(t)\} \tag{5.7}$$

and $T_2 = T_2(x_0, z_0; u_1, u_2, v_2)$ the instant of encounter of players P_2 and E_2 for strategies (u_1, u_2, v_2), if

$$T_2 = \min\{t \geqslant t_0 \mid x(t) = z(t)\} \tag{5.8}$$

If a time $t \geqslant t_0$ such that $x(t) = y(t)$ $(x(t) = z(t))$ does not exist, we assume $T_1(x_0, y_0; u_1, u_2, v_1)$ $(T_2(x_0, z_0; u_1, u_2, v_2))$ to be equal to $+\infty$. If T_1 and T_2 exist and are finite, we call the point $x(T_1) = y(T_1)$ $(x(T_2) = z(T_2))$ the point of encounter of players $P_1(P_2)$ with player $E_1(E_2)$ with strategies (u_1, u_2, v_1) $((u_1, u_2, v_2))$.

Player P_i is interested in encountering player E_i in the shortest time, i.e. he tries to minmize T_i. Player E_j tends to prolong the time of encounter with P_j.

Thus, the four-person on quick-action operation in standard form which we denote by the symbol $\Gamma(x_0, y_0, z_0)$ has been defined.

We will consider the case of cooperation between the pursuers and determine the characteristic set V of the cooperative game in conformity with the principle of constructing set (2.5). We shall call the set $P = \{P_1, P_2\}$ the coalition of pursuers. The strategy of coalition P is a vector function of the form $u = (u_1, u_2)$ whose components have values in the circles (5.2). The coalition aim is to minimize the components of the vector $T = (T_1, T_2)$.

We put $\alpha = \alpha_1 + \alpha_2$. From (5.1)–(5.5) it follows that if $\alpha \leqslant \beta_1$ $(\alpha \leqslant \beta_2)$, then $T_1 = +\infty$ $(T_2 = +\infty)$, since in that case the player $E_1(E_2)$ running away along the straight line that passes through points x_0 and $y_0(z_0)$ can always avoid an encounter with $P_1(P_2)$. We shall therefore assume that

$$\alpha > \max\{\beta_1, \beta_2\} \tag{5.9}$$

For any programmed control $v_j(t)$ of players E_j there exists a unique constant control

$\bar{u} = (\bar{u}_1, \bar{u}_2)$ of player P which guarantees to him an encounter with E_j in minimal time /6/. Such control prescribes to him the motion along a beam directed to the point of encounter. Control $\bar{u}$ is called the quick-action to the point of encounter with E_j. A parallel approach to player E_j (the Π_j–strategy) is called the method of pursuit by the coalition P of player E_j. The control of the coalition P is identical with the control which guarantees the quick-action operation in reaching the point of encounter with E_j. We denote the Π_j–strategy of coalition P by $u^{\Pi_j} = (u_1^{\Pi_j}, u_2^{\Pi_j})$.

We shall call the strategy $u^{\Pi} = (u_1^{\Pi}, u_2^{\Pi})$ of coalition P the Π–strategy, if it assigns to player P a parallel approach first with the player E_1 (E_2), and then with the player E_2 (E_1).

Note that the state of information in the game $\Gamma(x_0, y_0, z_0)$ allows the coalition P to use the Π-strategy.

We construct the characteristic sets $V(S, x_0, y_0, z_0)$, $S \subset P$. To construct the characteristic set $V(P_1, x_0, y_0)$ we shall consider the antagonistic game $\Gamma_{P_1/E_1}(x_0, y_0)$ between P_1 and E_1 in which the player P_1 aims at minimizing the time of encounter with E_1. By Theorem 5/ (/6/ p.27) there is an optimal Π–strategy for P_1 and an optimal programmed strategy for E_1

$$v_1^* = \beta_1 \frac{y_0 - x_0}{\| y_0 - x_0 \|}, \quad \| y_0 - x_0 \| = \rho(x_0, y_0) = [(x_0^1 - y_0^1)^2 - (x_0^2 - y_0^2)^2]^{1/2}$$

Since the strategy v_1^* assigns to player E_1 to run away from P_2 along a straight line which passes through the points x_0 and y_0, the optimal Π–strategy of player P_1 is the same as the strategy of linearized pursuit, i.e. the pursuit is along the straight line of escape of E_1. Therefore a generalized value exists (see Theorem 4 in /6/ p.25).

$$\text{val}\, \Gamma_{P_1/E_1}(x_0, y_0) = \frac{\rho(x_0, y_0)}{\alpha_1 - \alpha_2 - \beta_1}$$

We assume that $V(P_1, x_0, y_0) = \rho(x_0, y_0)/(\alpha_1 - \alpha_2 - \beta_1)$ (see (2.5)). Similarly we obtain $V(P_2, x_0, z_0) = \rho(x_0, z_0)/(\alpha_2 - \alpha_1 - \beta_2)$.

Let us calculate the characteristic set $V(P, x_0, y_0, z_0)$ on the assumption that the coalition P uses the Π–strategy. The sequence of pursuit of players E_1 and E_2 by coalition P is determined by two transformations $\pi_1 = \{E_1, E_2\}$, and $\pi_2 = \{E_2, E_1\}$, i.e. the set of all transformations in P is $\pi_P = \{\pi_1, \pi_2\}$. In carrying out the pursuit in accordance with the transformation π_1 the player E_2 moves along the straight line passing through the points s and z_0, where s is the point of encounter of P and E_1 (the optimal programmed strategy of player E_2 on the quick-action game $\Gamma_{P/E_2}(s, z_0)$). We denote that strategy of player E_2 by $\bar{v}_2(s)$. The locus of the points $\{s = (s^1, s^2)\}$ of encounter of P and E_1 in motion with velocities α and β_1, and using the coalition P is a circle /7/. The Π_1–strategy is a circle of Apollonius $A_1^{t_0}$:

$$\left(s^1 - \frac{\alpha^2 y_0^1 - \beta_1^2 x_0^1}{\alpha^2 - \beta_1^2}\right)^2 + \left(s^2 - \frac{\alpha^2 y_0^2 - \beta_1^2 x_0^2}{\alpha^2 - \beta_1^2}\right)^2 = \left(\frac{\alpha\beta_1}{\alpha^2 - \beta_1^2}\,\rho(x_0, y_0)\right)^2$$

Let us determine to which point $\bar{s}$ of the circle $A_1^{t_0}$ E_1 must move to obtain the maximum time of encounter with P. As shown by Shiryaev /6/ the point $\bar{s}$ is the solution of the problem

$$\max_{s \in A_1^{t_0}} (\| x_0 - s \| + \| s - z_0 \|)$$

Hence player E_1 under conditions of transformation π_1, must move in the direction of the point $\bar{s} = (\bar{s}^1, \bar{s}^2)$. We denote this strategy of E_1 by $\bar{v}_1$.

Lemma 2. If the order of pursuit of players E_1 and E_2 by the coalition P is determined by the transformation $\pi_1 = \{E_1, E_2\}$, then for any v_1 and v_2 from (5.5) the inequalities

$$\bar{T}^{E_1} = T^{E_1}(x_0, y_0; \bar{u}^{\Pi}, \bar{v}_1) \leqslant T^{E_1}(x_0, y_0; \bar{u}^{\Pi}, v_1)$$

$$\bar{T}^{E_2} = T^{E_2}(x_0, y_0, z_0; \bar{u}^{\Pi}, \bar{v}_1, \bar{v}_2) \leqslant T^{E_2}(x_0, y_0, z_0; \bar{u}^{\Pi}, \bar{v}_1, v_2)$$

are satisfied. Here $\bar{u}^{\Pi}$ is the Π–strategy of the coalition P which assigns to him a parallel approach, beginning with E_1 and, then, with E_2; $\bar{v}_2 = \bar{v}_2(\bar{s})$. $\bar{T}^{E_j}$ denotes the instant of encounter of the coalition with player E_j.

The proof of Lemma 2 is similar to the proof of the corresponding inequalities in /6, p. 61/.

Consider now the transformation π_2. By analogy we establish that player E_1 moves along the straight line passing through the points s and y_0 in the oppositve direction of s, where s is the encounter of P and E_2. We denote this strategy of player E_1 by $\bar{\bar{v}}_1(s)$). Player E_2 moves towards the point $\bar{\bar{s}} = (\bar{\bar{s}}^1, \bar{\bar{s}}^2)$ (the strategy $\bar{\bar{v}}_2$) of the circle of Apollonius $A_2^{t_0}$.

$$\left(\bar{\bar{s}}^1 - \frac{\alpha^2 z_0^1 - \beta_2^2 x_0^1}{\alpha^2 - \beta_2^2}\right)^2 + \left(\bar{\bar{s}}^2 - \frac{\alpha^2 z_0^2 - \beta_2^2 x_0^2}{\alpha^2 - \beta_2^2}\right)^2 = \left(\frac{\alpha\beta_2}{\alpha^2 - \beta_2^2}\,\rho(x_0, z_0)\right)^2$$

$$\max_{s \in A_2^{t_0}} (\| x_0 - s \| + \| s - y_0 \|) = \| x_0 - \bar{\bar{s}} \| + \| \bar{\bar{s}} - y_0 \|$$

Lemma 3. If the order of pursuit of players E_1 and E_2 by the coalition P is determined by the rearrangement $\pi_2 = \{E_2, E_1\}$, then, for any v_1 and v_2 from (5.5), the inequalities

$$\bar{\bar{T}}^{E_1} = T^{E_1}(x_0, y_0, z_0; \bar{\bar{u}}^{\Pi}, v_1, v_2) \leqslant T^{E_1}(x_0, y_0, z_0; \bar{\bar{u}}^{\Pi}, v_1, \bar{v}_2)$$

$$\bar{\bar{T}}^{E_2} = E^{E_2}(x_0, z_0; \bar{\bar{u}}^{\Pi}, \bar{v}_2) \leqslant T^{E_2}(x_0, z_0; \bar{\bar{u}}^{\Pi}, v_2)$$

are satisfied, where $\bar{\bar{u}}^{\Pi}$ is the Π-strategy of coalition P which assigns to him, first, a parallel approach to E_2 and, then to E_1; $\bar{v}_1 = \bar{v}_1(\bar{s})$. The instant of encounter of coalition P with player E_j is denoted by $\bar{\bar{T}}^{E_j}$.

Lemmas 2 and 3 show that the best means of escape for E_1 and E_2 are the following. With the transformation π_1 player E_1 moves to point $\bar{s} \in A_1^{t_0}$, while player E_2 moves to the diametrically opposite a side of $\bar{s}$ (Fig.1), and with trnasformation π_2 player E_2 moves to the point $\bar{\bar{s}} \in A_2^{t_0}$, and player E_1 to the diametrically opposite side from $\bar{\bar{s}}$ (Fig.1). Hence (see (2.5))

$$V(P, x_0, y_0, z_0) = \{(\bar{T}^{E_1}, \bar{T}^{E_2}), (\bar{\bar{T}}^{E_1}, \bar{\bar{T}}^{E_2})\}$$

Since the strategy of players which determine the characteristic sets, assigns to them rectilinear motions with constant velocities, the following representations hold:

$$\bar{T}^{E_1} = \frac{\rho(x_0, \bar{s})}{\alpha} = \frac{\rho(y_0, \bar{s})}{\beta_1}, \quad \bar{T}^{E_2} = \bar{T}^{E_1} + \frac{\rho(\bar{s}, z')}{\alpha - \beta_2}$$

$$\bar{\bar{T}}^{E_1} = \bar{\bar{T}}^{E_2} + \frac{\rho(\bar{\bar{s}}, y')}{\alpha - \beta_1}, \quad \bar{\bar{T}}^{E_2} = \frac{\rho(x_0, \bar{\bar{s}})}{\alpha} = \frac{\rho(z_0, \bar{\bar{s}})}{\beta_2}$$

In these z' (y') is the position of player E_2 (E_1) at the instant of encounter with P, and E_1 (E_2) at the point $\bar{s}$ $(\bar{\bar{s}})$ under conditions of transformation $\pi_1(\pi_2)$ (Fig.1). Since $\rho(\bar{s}, z') = \rho(\bar{s}, z_0) + \rho(z_0, z')$, $\rho(z_0, z') = \beta_2 \bar{T}^{E_1}$, and $\rho(\bar{\bar{s}}, \bar{\bar{y}}) = \rho(\bar{\bar{s}}, y_0) + \rho(y_0, \bar{\bar{y}})$, $\rho(y_0, \bar{\bar{y}}) = \beta_1 \bar{\bar{T}}^{E_2}$, we have

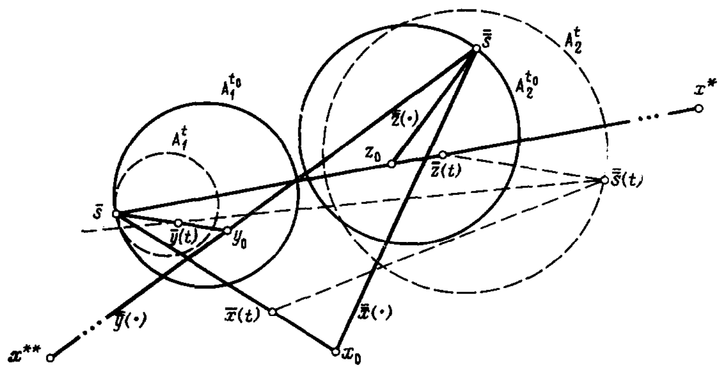

Fig.1

$$\bar{T}^{E_2} = \frac{\rho(x_0, \bar{s}) + \rho(\bar{s}, z_0)}{\alpha - \beta_2}, \quad \bar{\bar{T}}^{E_1} = \frac{\rho(x_0, \bar{\bar{s}}) + \rho(\bar{\bar{s}}, y_0)}{\alpha - \beta_1}$$

where $\bar{T}^{E_2}$ $(\bar{\bar{T}}^{E_1})$ is the time of encounter of P with E_2 (E_1), that corresponds to the transformation π_1 (π_2). Obviously

$$\bar{T}^{E_1} \leqslant \bar{\bar{T}}^{E_1}, \quad \bar{\bar{T}}^{E_2} \leqslant \bar{T}^{E_2} \tag{5.10}$$

We shall require the following inequalities to be satisfied:

$$\frac{\rho(x_0, y_0)}{\alpha_1 - \alpha_2 - \beta_1} \geqslant \frac{\rho(x_0, \bar{\bar{s}}) + \rho(\bar{\bar{s}}, y_0)}{\alpha - \beta_1} \tag{5.11}$$

$$\frac{\rho(x_0, z_0)}{\alpha_2 - \alpha_1 - \beta_2} \geqslant \frac{\rho(x_0, \bar{s}) + \rho(\bar{s}, z_0)}{\alpha - \beta_2} \tag{5.12}$$

Lemma 4. Let conditions (5.11) and (5.12) be satisfied. Then the set V is superadditive on $S \subset P$, i.e.

$$V(P, x_0, y_0, z_0) \supsetneqq V(P_1, x_0, y_0) \times V(P_2, x_0, z_0) \tag{5.13}$$

Proof. Since the set $V(P_1, x_0, y_0) \times V(P_2, x_0, z_0)$ consists of a single vector

$$\eta = (\eta_1, \eta_2) = \left(\frac{\rho(x_0, y_0)}{\alpha_1 - \alpha_2 - \beta_1}, \frac{\rho(x_0, z_0)}{\alpha_2 - \alpha_1 - \beta_2} \right)$$

to prove the lemma it is sufficient to show the validity of at least one of the two pairs of inequalities

$$T^{E_j} \leqslant \eta_j, \quad j = 1, 2 \tag{5.14}$$

$$\overline{T}^{E_j} \leqslant \eta_j, \quad j = 1, 2 \tag{5.15}$$

It is evident that inequality (5.14) for $j = 1$ follows from (5.10) and (5.11) and for $j = 2$ from (5.12). We can also show the validity of inequalities (5.15).

Theorem 3. If conditions (5.11) and (5.12) are satisfied, then the vectors $\bar{\xi} = (\overline{T}^{E_1}, \overline{T}^{E_2})$, $\bar{\bar{\xi}} = (\overline{\overline{T}}^{E_1}, \overline{\overline{T}}^{E_2})$ represent sharings and belong to the c-kernel $C_V(x_0, y_0, z_0)$ of the cooperative game $\Gamma^V(x_0, y_0, z_0)$. There are no other sharings in the game $\Gamma_V(x_0, y_0, z_0)$.

Proof. Since $\bar{\xi}, \bar{\bar{\xi}} \in V(P, x_0, y_0, z_0)$ from the inequalities (5.14) and (5.15) it follows that the vectors $\bar{\xi}$ and $\bar{\bar{\xi}}$ are sharings in the game $\Gamma_V(x_0, y_0, z_0)$. There are no other such vectors. The inequality (5.10) implies that the sharings $\bar{\xi}$ and $\bar{\bar{\xi}}$ do not dominate one other, i.e. they belong to the c-kernel $C_V(x_0, y_0, z_0)$.

Definition 8. The trajectory $\bar{x}(\cdot) = x(\cdot, x_0, \bar{u}^{\Pi})$ $(\bar{\bar{x}}(\cdot) = x(\cdot, x_0, \bar{\bar{u}}^{\Pi}))$ of system (5.1) which corresponds to strategy $\bar{u}^{\Pi}$ $(\bar{\bar{u}}^{\Pi})$ of coalition P, i.e. $\bar{x}(t_0) = x_0$, $\bar{x}(\overline{T}^{E_1}) = \bar{s}$, $\bar{x}(\overline{T}^{E_2}) = x^*$ $(\bar{\bar{x}}(t_0) = x_0$, $\bar{\bar{x}}(\overline{\overline{T}}^{E_2}) = \bar{\bar{s}}$, $\bar{\bar{x}}(\overline{\overline{T}}^{E_1}) = x^{**})$, where x^* (x^{**}) is the point of encounter between P and E_2 (E_1) under conditions of a π_1 (π_2) transformation, when E_2 (E_1) uses the strategy $\bar{v}_2 = \bar{v}_2(\bar{s})$ $(\bar{v}_1 = \bar{v}_1(\bar{\bar{s}}))$. In Fig.1 that trajectory is represented by the broken line $x_0\bar{s}x^*$ $(x_0\bar{\bar{x}}x^{**})$, it is called the conditionally optimal trajectory.

Definition 9. If in the motion along the trajectory $\bar{x}(\cdot)(\bar{\bar{x}}(\cdot))$ the sharing $\bar{\xi}$ $(\bar{\bar{\xi}})$ is dynamically stable (in the sense of Defintion 5), the trajectory $\bar{x}(\cdot)$ $(\bar{\bar{x}}(\cdot))$ is called the optimal trajectory.

We shall denote the trajectory of system (5.3), (5.4) which corresponds to strategy $\bar{v}_1(\bar{v}_2)$ of player E_1 (E_2) used by him under conditions of transformation π_1 by $\bar{y}(\cdot) = y(\cdot, y_0, \bar{v}_1)$ $(\bar{z}(\cdot) = z(\cdot, z_0, \bar{v}_2))$, i.e. $\bar{y}(t_0) = y_0$, $\bar{y}(\overline{T}^{E_1}) = \bar{s}$ $(\bar{z}(t_0) = z_0$, $\bar{z}(\overline{T}^{E_2}) = x^*)$. In Fig.1 the trajectory $\bar{y}(\cdot)$ $(\bar{z}(\cdot))$ is represented by the segment $y_0\bar{s}$ (z_0x^*).

Let $t \in [t_0, \overline{T}^{E_1})$. Consider the game $\Gamma_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ beginning from the state $\bar{x}(t), \bar{y}(t), \bar{z}(t)$. By definition

$$V(P_1, \bar{x}(t), \bar{y}(t)) = \frac{\rho(\bar{x}(t), \bar{y}(t))}{\alpha_1 - \alpha_2 - \beta_1}$$

$$V(P_2, \bar{x}(t), \bar{z}(t)) = \frac{\rho(\bar{x}(t)), \bar{z}(t))}{\alpha_2 - \alpha_1 - \beta_2}$$

As follows from the definition of the Π-strategy $\bar{u}^{\Pi}$, the segment connecting point $\bar{x}(t)$ and $\bar{y}(t)$, is parallel to the segment that connects the point x_0 and y_0 for all $t \in [t_0, \overline{T}^{E_1})$. Consequently the locus of the points of encounter of P and E_1 (the circle of Apollonius A_1^t) in the game $\Gamma_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ is contained inside the circle $A_1^{t_0}$ and touches it as the points $\bar{s}$. Therefore

$$V(P, \bar{x}(t), \bar{y}(t), z(t)) = \{\bar{\xi}^t, \bar{\eta}^t\}$$

$$\bar{\xi}^t = (\bar{\xi}_1{}^t, \xi_2{}^t) = \left(\frac{\rho(\bar{y}(t), \bar{s})}{\beta_1}, \frac{\rho(\bar{x}(t), \bar{s}) + \rho(\bar{s}, z(t))}{\alpha - \beta_2}\right)$$

$$\bar{\eta}^t = (\bar{\eta}_1{}^t, \bar{\eta}_2{}^t) = \left(\frac{\rho(\bar{x}(t), \bar{\bar{s}}(t)) + \rho(\bar{\bar{s}}(t), \bar{y}(t))}{\alpha - \beta_1}, \frac{\rho(\bar{z}(t), \bar{\bar{s}}(t))}{\beta_2}\right)$$

where $\bar{\bar{s}}(t)$ is a point on the Apollonius circle $A_2{}^t$ such that

$$\max_{s(t) \in \overset{+}{A_2}} (\| \bar{x}(t) - s(t) \| + \| s(t) - \bar{y}(t) \|) = \| \bar{x}(t) - \bar{\bar{s}}(t) \| + \| \bar{\bar{s}}(t) - \bar{y}(t) \|$$

where $\bar{\xi}_1{}^t$ $(\bar{\xi}_2{}^t)$ is the time of encounter between P and E_1 (E_2) in the game $\Gamma_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ under conditions of transformation π_1, and $\bar{\eta}_1{}^t$ $(\bar{\eta}_2{}^t)$ is the time of encounter between P and E_1 (E_2) under conditions of transformation π_2, Obviously

$$\bar{\xi}_1{}^t \leqslant \bar{\eta}_1{}^t, \quad \bar{\eta}_2{}^t \leqslant \bar{\xi}_2{}^t \tag{5.16}$$

We shall require that the following inequality shall be satisfied

$$\frac{\rho(\bar{x}(t), \bar{z}(t))}{\alpha_2 - \alpha_1 - \beta_2} \geqslant \frac{\rho(\bar{x}(t), \bar{s}) + \rho(\bar{s}, \bar{z}(t))}{\alpha - \beta_2} \tag{5.17}$$

We denote by $\hat{y}$ the points of encounter of P_1 and E_1 in the game $\Gamma_{P_1/E_1}(\bar{x}(t), \bar{y}(t))$. Then

$$\frac{\rho(\bar{x}(t), \bar{y}(t))}{\alpha_1 - \alpha_2 - \beta_1} = \frac{\rho(\bar{x}(t), y^{\wedge})}{\alpha_1 - \alpha_2}$$

Since the player E_1 in the game $\Gamma_{P_1/E_1}(\bar{x}(t), \bar{y}(t))$ attains the maximum time of encounter with P, running along the straight line that passes through the points $\bar{x}(t)$ and $\bar{y}(t)$ at constant velocity β_1 hence $\rho(\bar{y}(t), \hat{y}) \geqslant \rho(\bar{y}(t), \bar{s})$ (otherwise E_1 would run in the direction of the point $\bar{s}$). This implies that $\rho(\bar{x}(t), \hat{y}) \geqslant \rho(\bar{x}(t), \bar{s})$. Consequently

$$\frac{1}{\alpha}\rho(\bar{x}(t), \bar{s}) \leqslant \frac{1}{\alpha}\rho(\bar{x}(t), y^{\wedge}) \leqslant \frac{1}{\alpha - \alpha_2}\rho(\bar{x}(t), y^{\wedge})$$

Since

$$\frac{1}{\alpha}\rho(\bar{x}(t), \bar{s}) = \frac{1}{\beta_1}\rho(\bar{y}(t), \bar{s})$$

we obtain

$$\frac{\rho(\bar{x}(t), \bar{y}(t))}{\alpha_1 - \alpha_2 - \beta_1} \geqslant \frac{\rho(\bar{y}(t), \bar{s})}{\beta_1} \tag{5.18}$$

The inequalities (5.17) and (5.18) show that the vector $\bar{\xi}^t$ is a sharing in the game $\Gamma_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$. From inequalities (5.16) it follows that the sharing is not dominated, i.e.

$$\bar{\xi}^t \in C_V(\bar{x}(t), \bar{y}(t), \bar{z}(t)), \quad t_0 \leqslant t < \bar{T}^{E_1} \tag{5.19}$$

Consider now the segment $[\bar{T}^{E_1}, \bar{T}^{E_2}]$ on which the vector $\bar{\eta}^t$ is not defined, since player E_1 is already caught by the coalition P (at the instant $\bar{T}^{E_1}$ at the point $\bar{s}$) hence the pursuit of the escapers in accordance with the transformation π_2 makes no sense. Since at $t \in [\bar{T}^{E_1}, \bar{T}^{E_2}]$ we assume that $\rho(\bar{x}(t), \bar{s}) = \rho(\bar{y}(t), \bar{s}) = 0$, hence the vector

$$\bar{\xi}^t = \left(0, \frac{\rho(\bar{x}(t), \bar{z}(t))}{\alpha - \beta_2}\right)$$

is a single sharing in the game $\Gamma_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ and consequently,

$$\bar{\xi}^t \in C_V(\bar{x}(t), \bar{y}(t), \bar{z}(t)), \quad \bar{T}^{E_1} \leqslant t \leqslant \bar{T}^{E_2} \tag{5.20}$$

The sharing $\bar{\xi} \in C_V(x_0, y_0, z_0)$ may be represented as follows:

$$\bar{\xi}_i = \begin{cases} t + \bar{\xi}_i^t, & t_0 \leqslant t \leqslant \bar{T}^{E_1} \\ \tau_i(t) + \bar{\xi}_i^t, & \bar{T}^{E_1} < t \leqslant \bar{T}^{E_2}; \quad i = 1, 2 \end{cases} \tag{5.21}$$

$$\tau_1(t) = \bar{T}^{E_1}, \quad \tau_2(t) = t, \quad \bar{\xi}^t = (\bar{\xi}_1^t, \bar{\xi}_2^t) \in C_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))$$

From (5.19)–(5.21) we have

$$\bar{\xi} \in \bigcap_{t_0 \leqslant t \leqslant \bar{T}^{E_2}} [\tau(t) + C_V(\bar{x}(t), \bar{y}(t), \bar{z}(t))]$$

$$\tau(t) = (\tau_1(t), \tau_2(t)), \quad \tau_1(t) = \min\{\bar{T}^{E_1}, t\}, \quad \tau_2(t) = t$$

Hence along the conditionally optimal trajectory $\bar{x}(\cdot)$ the sharing $\bar{\xi}$ belonging to the c-kernel $C_V(x_0, y_0, z_0)$ is dynamically stable.

Consider instead of $\bar{x}(\cdot)$ the conditionally optimal trajectory $\bar{\bar{x}}(\cdot)$ assuming that in the half-interval $[t_0, \bar{\bar{T}}^{E_2})$

$$\frac{\rho(\bar{\bar{x}}(t), \bar{\bar{y}}(t))}{\alpha_1 - \alpha_2 - \beta_1} \geqslant \frac{\rho(\bar{\bar{x}}(t), \bar{\bar{s}}) + \rho(\check{s}, \bar{\bar{y}}(t))}{\alpha - \beta_1} \tag{5.22}$$

(here $\bar{\bar{z}}(\cdot)$ ($\bar{\bar{y}}(\cdot)$) are the trajectories of system (5.4), (5.3)) corresponding to the strategy $\bar{v}_2$ ($\bar{v}_1$) of player E_2 (E_1), applicable under conditions of the transformation π_2. Similarly we have

$$\bar{\bar{\xi}} \in \bigcap_{t_0 \leqslant t \leqslant \bar{\bar{T}}^{E_1}} [\theta(t) + C_V(\bar{\bar{x}}(t), \bar{\bar{y}}(t), \bar{\bar{z}}(t))]$$

$$\theta(t) = (\theta_1(t), \theta_2(t)), \quad \theta_1(t) = t, \quad \theta_2(t) = \min\{\bar{\bar{T}}^{E_2}, t\}$$

Thus the theorem about the dynamic stability of the c-kernel in the game $\Gamma_V(x_0, y_0, z_0)$ is valid.

Theorem 4. In the half-interval $[t_0, \bar{T}^{E_1})$ suppose condition (5.17) and in the half-interval $[t_0, \bar{\bar{T}}^{E_2})$ condition (5.22) is satisfied.

Then in the game $\Gamma_V(x_0, y_0, z_0)$ a dynamically stable c-kernel $C_V(x_0, y_0, z_0) = \{(\bar{T}^{E_1}, \bar{T}^{E_2}), (\bar{\bar{T}}^{E_1}, \bar{\bar{T}}^{E_2})\}$ exists (in the meaning of Definition 5).

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THE USE OF FIRST INTEGRALS IN PROBLEMS OF SYNTHESIZING OPTIMAL CONTROL SYSTEMS*

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The problem of synthesizing optimal control of the motion of a non-linear unsteady system is considered. The control quality is evaluated by a functional of mixed type (a Boltz functional) /1/. A method of synthesizing optimal control systems is worked out for systems of variational problems with a fixed time and a free right end, based on the use of first integrals of the equations of a free uncontrolled object. The effectiveness of the proposed method is illustrated by examples. The synthesis problem, i.e. of representing the optimal control as a function of the system coordinates, has been considered in many publications, for instance in /1-9/ etc.

1. Consider a controllable object whose motion is defined by the equations

$$x^{\cdot} = f(x, t) + b(x, t)\, u(x, t) \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$ is an n-dimensional vector of the phase coordinates, a dot denotes differentiation with respect to t; $u = (u_1, \ldots, u_r)$ is an r-dimensional vector of the controlling functions, $f = (f_1, \ldots, f_n)$, $b = (b_{ij})$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$ are an n-dimensional vector function, and an $n \times r$ functional matrix respectively specified on some open set Ω of Euclidean space E_{n+1}, in which the coordinates of a point are the numbers $x_1, \ldots, x_n, t$. Henceforth we assume that f, b, u are such that function $f_* = f(x, t) + b(x, t)\, u(x, t)$ and its partial derivatives $\partial f_*/\partial x_i$ $(i = 1, 2, \ldots, n)$ exist and are continuous in the open set Ω.

We call the arbitrary function $u(x, t)$ that satisfies the conditions on $f_*(x, t)$ with values in the Euclidean space E_r the admissible control.

Suppose we are given t_1, t_2, the instants of the beginning and end of the control process and let the initial state of the object be

$$x(t_1) = x_a \tag{1.2}$$

We denote by $v_1(x, t), \ldots, v_k(x, t)$, $k \leqslant n$ the independent first integrals /10/ of the equations of motion of the free (uncontrolled) object, i.e. of the system of equations

$$x^{\cdot} = f(x, t) \tag{1.3}$$

Let $W(y_1, \ldots, y_k)$ be a given arbitrary differentiable function. We select as the arguments y_m the first integrals $v_m(x, t)$, and consider the functional

$$\Phi = W\{v[x(t_2), t_2]\} + \frac{1}{2}\int_{t_1}^{t_2}\sum_{j=1}^{r}\left\{k_j\sum_{i=1}^{n}\frac{\partial W[v(x, t)]}{\partial x_i}\, b_{ij}(x, t)\right\}^2 dt + \tag{1.4}$$
$$\frac{1}{2}\int_{t_1}^{t_2}\sum_{j=1}^{r}\left[\frac{u_j(x, t)}{k_j}\right]^2 dt$$

where $v(x, t) = \{v_1(x, t), \ldots, v_k(x, t)\}$ is the vector of first integrals and $k_1, \ldots, k_r$ are specified coefficients.

The first term of the functional (1.4) (the terminal part) is a function of the phase coordinates at the end of the control process and of finite instant of time t_2, the second defines the properties of the object itself as well as its control system. The third term of the functional Φ can be interpreted as the costs of controlling the motion of the object /9/.

The physical meaning of the first two terms of the quality criterion (1.4) can be revealed by the specific selection of the function W and the first integrals $v_m(x, t)$. For example, when

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